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Systems of Covariance in Quantum Probability

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Symmetry groups and systems of covariance are investigated in the framework of quantum probability theory. It is shown that a measurement X can be represented by a positive operator-valued measure $\hat{X}^S$ on a sector S of the amplitude space. Moreover, $\hat{X}^S$ provides a generalized system of covariance for the generalized unitary representation of a symmetry group.

1. INTRODUCTION

The author has recently developed a quantum probability theory (Gudder, 1988, 1989, 1990) based on ideas due to Feynman (1948; Feynman and Hibbs, 1965). The main principle in this framework is that the amplitude of a measurement outcome x is the "sum" of the amplitudes of the alternatives (or samples) that result in x and the probability that x occurs is the absolute value squared of its amplitude. I first review the mathematical formulation of this principle. The basic concepts of this formulation are measurements and amplitudes. These are defined as certain functions on the sample space Ω. I define a superposition relation on the set of amplitudes and construct sectors in the amplitude space. These sectors correspond to superselection sectors for a physical system. I then show that a measurement X can be represented by a positive operator-valued measure $\hat{X}^S$ on an arbitrary sector S.

The remainder of the paper is devoted to the study of symmetry groups. A symmetry group G is defined as a group of bijections on the sample space Ω that preserve the measurement structure. A symmetry group induces a generalized unitary representation U_g, $g \in G$, on the amplitude space in a natural way. The unitary transformations U_g, $g \in G$, then map a sector S onto other sectors $U_g S$. The main result shows that $\hat{X}^S$ provides a generalized

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system of covariance for the generalized unitary representation U_g. It is shown that if G leaves sectors invariant, then $\hat{X}^S$ gives an ordinary system of covariance for the unitary representation U_g.

2. PRELIMINARY RESULTS

This section reviews some of the basic principles of quantum probability theory and presents various preliminary results that will be needed in the sequel. For further motivation and details see Gudder (1988, 1989, 1990).

Let Ω be a nonempty set which we call a *sample space* and whose elements we call *sample points*. A surjection $X: \Omega \to R(X)$ is a *measurement* if the following conditions hold.

(M1) $R(X)$ is the base space of a measure space $(R(X), \Sigma_X, \mu_X)$.
(M2) For every $x \in R(X)$, $X^{-1}(x)$ is the base space of a measure space $(X^{-1}(x), \Sigma_X^x, \mu_X^x)$.

We call the elements of $R(X)$ *X-outcomes*, the sets in Σ_X *X-events*, and $X^{-1}(x)$ the *fiber* (or *sample*) *over* x. We call $H_X = L^2(R(X), \Sigma_X, \mu_X)$ the *Hilbert space* for X. Denote by $\hat{\mathscr{A}}(\Omega)$ the set of all measurements on Ω. A subset $\mathscr{A} \subseteq \hat{\mathscr{A}}(\Omega)$ is called a *catalog* if for any ω, $\omega' \in \Omega$ there exists an $X \in \mathscr{A}$ such that $X(\omega) \neq X(\omega')$. A function $f: \Omega \to \mathbb{C}$ is an *amplitude* for a catalog $\mathscr{A}$ if the following conditions hold.

(A1) $f|X^{-1}(x) \in L^1(X^{-1}(x), \Sigma_X^x, \mu_X^x)$ for every $x \in R(X)$ and $X \in \mathscr{A}$.
(A2) $f_X \equiv \int f\, d\mu_X^x \in H_X$ for all $X \in \mathscr{A}$.
(A3) $\|f_X\| = \|f_Y\|$ for every X, $Y \in \mathscr{A}$.

We denote the set of amplitudes for $\mathscr{A}$ by $\mathscr{H}(A)$ and call $\mathscr{H}(A)$ the *amplitude space* for $\mathscr{A}$. For $f \in \mathscr{H}(\mathscr{A})$ we write $\|f\| = \|f_X\|$, where $X \in \mathscr{A}$ is arbitrary, and if $\|f\| = 1$, we call f an *amplitude density*. Moreover, we denote the set of amplitude densities by $\mathscr{D}(\mathscr{A})$. Notice that if $f \in \mathscr{H}(\mathscr{A})$, $a \in \mathbb{C}$, then $af \in \mathscr{H}(\mathscr{A})$ and $\|af\| = |a|\,\|f\|$. Also, if $f \in \mathscr{H}(\mathscr{A})$ with $\|f\| \neq 0$, then $f/\|f\| \in \mathscr{D}(\mathscr{A})$.

If $f \in \mathscr{D}(\mathscr{A})$, we call f_X the *(X, f)-wave function*. We interpret $f(\omega)$ as the amplitude of the sample point ω and $f_X(x)$ as the probability amplitude of the X-outcome x. The probability density at x is then given by $|f_X(x)|^2$. We define the *(X, f)-probability* of an X-event A by

$$P_{X,f}(A) = \int_A |f_X|^2\, d\mu_X$$

Notice that $P_{X,f}$ is a probability measure on Σ_X which we call the *f-distribution* of X.

For $f, g \in \mathscr{H}(\mathscr{A})$ we write $f s g$ if for every X, $Y \in \mathscr{A}$ we have

$$\int f_X \bar{g}_X \, d\mu_X = \int f_Y \bar{g}_Y \, d\mu_Y \tag{2.1}$$

If (2.1) holds, we denote this expression by $\langle f, g \rangle$. Notice that s is a reflexive, symmetric relation and if $f \; s \; g$, then $af \; s \; g$ for all $a \in \mathbb{C}$. We call s the *superposition relation.*

Theorem 2.1. Let $f, g \in \mathscr{H}(\mathscr{A})$. Then $f s g$ if and only if $f+g$,

$$f + ig \in \mathscr{H}(\mathscr{A})$$

Proof. Suppose $f s g$. Then $f+g$ clearly satisfies (A1) and (A2). Moreover,

$$\int |(f+g)_X|^2 \, d\mu_X = \int |(f_X + g_X)|^2 \, d\mu_X$$

$$= \|f\|^2 + \|g\|^2 + 2 \operatorname{Re} \int f_X \bar{g}_X \, d\mu_X \tag{2.2}$$

so condition (A3) holds. Hence, $f+g \in \mathscr{H}(\mathscr{A})$. Since $f s (ig)$, it follows that $f + ig \in \mathscr{H}(\mathscr{A})$. Conversely, if $f+g \in \mathscr{H}(\mathscr{A})$, then from (2.2) we have

$$\operatorname{Re} \int f_X \bar{g}_X \, d\mu_X = \operatorname{Re} \int f_Y \bar{g}_Y \, d\mu_Y$$

for every X, $Y \in \mathscr{A}$. If, in addition, $f + ig \in \mathscr{H}(\mathscr{A})$, then since

$$\int |(f+ig)_X|^2 \, d\mu_X = \int |(f_X + ig_X)|^2 \, d\mu_X$$

$$= \|f\|^2 + \|g\|^2 + 2 \operatorname{Im} \int f_X \bar{g}_X \, d\mu_X$$

we have

$$\operatorname{Im} \int f_X \bar{g}_X \, d\mu_X = \operatorname{Im} \int f_Y \bar{g}_Y \, d\mu_Y$$

for every X, $Y \in \mathscr{A}$. It follows that $f s g$. ■

Corollary 2.2. For $f, g \in \mathscr{H}(\mathscr{A})$, $f\, s\, g$ if and only if $af+bg \in \mathscr{H}(\mathscr{A})$ for every $a, b \in \mathbb{C}$.

For $B \subseteq \mathscr{H}(\mathscr{A})$ we write

$$B^s = \{f \in \mathscr{H}(\mathscr{A}) : f\, s\, g \text{ for all } g \in B\}$$

We call $B \subseteq \mathscr{H}(\mathscr{A})$ an *s-set* if $B \subseteq B^s$. Thus, B is an s-set if and only if $f\, s\, g$ for every $f, g \in B$. It is clear that singleton sets are s-sets and hence every $f \in \mathscr{H}(\mathscr{A})$ is contained in an s-set. Moreover, by Zorn's lemma, every s-set is contained in a maximal s-set. We denote the collection of maximal s-sets by $\mathscr{M}(\mathscr{A})$. An element of $\mathscr{M}(\mathscr{A})$ is a maximal set of amplitudes for which superpositions are allowed. They correspond to superselection sectors for a physical system. Let $M \in \mathscr{M}(\mathscr{A})$. If $f, g \in M$, $a, b \in \mathbb{C}$, then by Corollary 2.2, $af+bg \in \mathscr{H}(\mathscr{A})$. Also, it is clear that $(af+bg)\, s\, h$ for every $h \in M$. Since M is maximal, $af+bg \in M$. Hence, M is a linear space. We call $f \in \mathscr{H}(\mathscr{A})$ a *null amplitude* if $\|f\| = 0$. It is clear that the set of null amplitudes forms a subspace of every $M \in \mathscr{M}(\mathscr{A})$. If we identify amplitudes that differ by a null amplitude, it is straightforward to show that $\langle \cdot, \cdot \rangle$ is an inner product on M. The Hilbert space formed by completing M relative to this inner product is called the *sector generated* by M. The collection of all sectors is denoted $\mathscr{S}(\mathscr{A})$. In general, $\mathscr{A}$ can have many sectors (Gudder, to appear).

In the sequel, S will denote a fixed sector generated by $M \in \mathscr{M}(\mathscr{A})$. For $X \in \mathscr{A}$, define $U_X^M : M \to H_X$ by $U_X^M f = f_X$. Then U_X^M is a linear transformation satisfying

$$\langle U_X^M f, U_X^M g \rangle = \langle f, g \rangle \tag{2.3}$$

for all $f, g \in M$. Since M is dense in S, there exists a unique linear extension U_X^S of U_X^M. It follows from (2.3) that $U_X^S : S \to H_X$ is a unitary transformation and its range $U_X^S S$ is a closed subspace of H_X. Let P_X^S be the orthogonal projection of H_X onto $U_X^S S$ and define $V_X^S : H_X \to S$ by $V_X^S = (U_X^S)^{-1} P_X^S$. Notice that $U_X^S V_X^S = P_X^S$ and $V_X^S U_X^S = I$. For $A \in \Sigma_X$ define $\hat{X}^S(A) : S \to S$ by $\hat{X}^S(A) = V_X^S \chi_A U_X^S$, where χ_A is the characteristic function projection $\chi_A h(x) = \chi_A(x) h(x)$. Clearly, $\hat{X}^S(A)$ is a bounded linear operator. Moreover, $\hat{X}^S(A)$ is positive since

$$\begin{aligned}\langle \hat{X}^S(A) f, f \rangle &= \langle V_X^S \chi_A U_X^S f, f \rangle = \langle P_X^S \chi_A U_X^S f, U_X^S f \rangle \\ &= \langle \chi_A U_X^S f, U_X^S f \rangle = \|\chi_A U_X^S f\|^2 \geq 0\end{aligned} \tag{2.4}$$

for all $f \in S$. We also obtain from (2.4) that

$$\langle \hat{X}^S(A) f, f \rangle \leq \|f\|^2$$

so $0 \le \hat{X}^S \le I$. If $f \in \mathscr{D}(\mathscr{A}) \cap S$, then applying (2.4) gives

$$\langle \hat{X}^S(A)f, f \rangle = \|\chi_A f_X\|^2 = P_{X,f}(A)$$

so $\hat{X}^S$ determines the f-distribution of X. Finally, $A \mapsto \hat{X}^S(A)$ is a positive operator-valued (POV) measure from Σ_X to S. Indeed, $\hat{X}^S(R(X)) = I$ and if $A_i \in \Sigma_X$ are mutually disjoint, then

$$\hat{X}^S(\cup A_i) = V_X^S \chi_{\cup A_i} U_X^S = V_X^S \sum \chi_{A_i} U_X^S = \sum V_X^S \chi_{A_i} U_X^S = \sum \hat{X}^S(A_i)$$

where convergence is in the strong operator topology. We conclude that every $X \in \mathscr{A}$ can be represented by a POV measure from Σ_X to S.

3. SYMMETRY GROUPS

If g_1, g_2 are bijections from Ω onto Ω, we denote their composition $g_1 \circ g_2$ simply by $g_1 g_2$. Under this operation, the set of all bijections $\hat{G}$ becomes a group. The identity $e \in \hat{G}$ is the identity function. If $g \in \hat{G}$, $B \subseteq \Omega$, we use the notation

$$gB = \{g(\omega) : \omega \in B\}$$

For a catalog $\mathscr{A} \subseteq \hat{\mathscr{A}}(\Omega)$, we say that a subgroup $G \subseteq \hat{G}$ *preserves fibers* if for every $X \in \mathscr{A}$, $x \in R(X)$, $g \in G$, $gX^{-1}(x)$ is an X-fiber. Clearly, if G preserves fibers, then for fixed $g \in G$, the map $X^{-1}(x) \mapsto gX^{-1}(x)$ is a bijection on the set of X-fibers. Moreover, for $x \in R(X)$, $g \in G$ there exists a unique $x_g \in R(X)$ such that $gX^{-1}(x) = X^{-1}(x_g)$. Hence, $x \mapsto x_g$ is a function from $R(X)$ into $R(X)$.

Lemma 3.1. If G preserves fibers, then $x \mapsto x_g$ is a bijection on $R(X)$ for every $X \in \mathscr{A}$, $g \in G$ and $x_{g_1 g_2} = (x_{g_2})_{g_1}$ for every $g_1, g_2 \in G$.

Proof. To show that $x \mapsto x_g$ is injective, suppose $x_g = x'_g$. Then $gX^{-1}(x) = gX^{-1}(x')$, so $X^{-1}(x) = X^{-1}(x')$ and $x = x'$. To show that $x \mapsto x_g$ is surjective, let $x \in R(X)$. Then $g^{-1}X^{-1}(x) = X^{-1}(x_{g^{-1}})$. Hence

$$X^{-1}(x) = gX^{-1}(x_{g^{-1}}) = X^{-1}[(x_{g^{-1}})_g]$$

Therefore, $x = (x_{g^{-1}})_g$. Finally, for $g_1, g_2 \in G$ we have

$$X^{-1}[(x_{g_2})_{g_1}] = g_1 X^{-1}(x_{g_2}) = g_1 g_2 X^{-1}(x) = X^{-1}(x_{g_1 g_2})$$

Hence, $x_{g_1 g_2} = (x_{g_2})_{g_1}$. ■

If G preserves fibers, $g \in G$, and $A \subseteq R(X)$, we use the notation

$$gA = \{x_g : x \in A\}$$

We say that G *preserves events* if for every $X\in\mathscr{A}$, $g\in G$, $A\mapsto gA$ is a bijection on Σ_X. If G preserves fibers and events, then G is a *symmetry group* on $\mathscr{A}$ if for every $X\in\mathscr{A}$ we have:

(S1) $g|X^{-1}(x)$ map Σ_X^x onto $\Sigma_X^{x_g}$ and $\mu_X^{x_g}(gB)=\mu_X^x(B)$ for every $x\in R(X)$ and $B\in\Sigma_X^x$.

(S2) $\mu_X(gA)=\mu_X(A)$ for all $A\in\Sigma_X$.

In the sequel, G will denote a symmetry group on $\mathscr{A}$. We interpret G as a group of bijections that preserves the measurement structure. For $f\in\mathscr{H}(\mathscr{A})$, $g\in G$, define $U_gf\colon\Omega\to\mathbb{C}$ by $U_gf(\omega)=f(g^{-1}\omega)$. Notice that $U_e=I$.

Theorem 3.2. (a) The map U_g is a bijection on $\mathscr{H}(\mathscr{A})$ satisfying $U_{g_1g_2}=U_{g_1}U_{g_2}$ for every $g_1, g_2\in G$. (b) If $M\in\mathscr{M}(\mathscr{A})$ then $U_gM\in\mathscr{M}(\mathscr{A})$ and U_g is a unitary transformation from M onto U_gM.

Proof. (a) For $f\in\mathscr{H}(\mathscr{A})$, $X\in\mathscr{A}$, U_gf satisfies (A1), and by (S1) we have

$$\begin{aligned}(U_gf)_X&=\int_{X^{-1}(x)} U_gf(\omega)\,d\mu_X^x(\omega)\\ &=\int_{X^{-1}(x)} f(g^{-1}\omega)\,d\mu_X^x(\omega)\\ &=\int_{X^{-1}(x_g-1)} f(g^{-1}\omega)\,d\mu_X^{x_g-1}(g^{-1}\omega)\\ &=f_X(x_{g^{-1}})\end{aligned}\tag{3.1}$$

Hence, applying (S2) gives

$$\|(U_gf)x\|^2=\int|f_X(x_{g^{-1}})|^2\,d\mu_X(x)=\int|f_X(x)|^2\,d\mu(x)=\|f_X\|^2$$

Thus, $U_gf\in\mathscr{H}(\mathscr{A})$. It is clear that U_g is injective. To show that U_g is surjective, suppose $h\in\mathscr{H}(\mathscr{A})$. Define $f\colon\Omega\to\mathbb{C}$ by $f(\omega)=h(g\omega)$. Then $f\in\mathscr{H}(\mathscr{A})$ and $U_gf=h$. Finally, we have for $f\in\mathscr{H}(\mathscr{A})$

$$U_{g_1g_2}f(\omega)=f(g_2^{-1}g_1^{-1}\omega)=U_{g_2}f(g_1^{-1}\omega)=U_{g_1}U_{g_2}f(\omega)$$

(b) If $f\,s\,h$, then by (S2) and (3.1) we have

$$\int(U_gf)_X\overline{(U_gh)}_X\,d\mu_X\int f_X(x_{g^{-1}})\bar h_X(x_{g^{-1}})\,d\mu_X(x)=\int f_X\bar h_X\,d\mu_X$$

Hence, $U_gf\,s\,U_gh$. It easily follows that $U_gM\in\mathscr{M}(\mathscr{A})$. This also shows that U_g is a unitary transformation from M onto U_gM. ■

We interpret $U_g f$ as the amplitude corresponding to f after the system has been transformed by the symmetry g. For $M \in \mathcal{M}(\mathcal{A})$ we write $gM = U_g M \in \mathcal{M}(\mathcal{A})$. Let S be the sector generated by M and gS the sector generated by gM. Since U_g is a unitary transformation from M onto gM, U_g has a unique unitary extension, which we also denote by U_g, from S onto gS. Since $U_g\colon S \to gS$ is a unitary tansformation satisfying $U_{g_1 g_2} = U_{g_1} U_{g_2}$, we call $g \mapsto U_g$ a *generalized unitary representation* of G. We use the adjective "generalized" since U_g can map S onto another Hilbert space. For $X \in \mathcal{A}$, $h \in H_X$, we define $\hat{U}_g h(x) = h(x_{g^{-1}})$ for every $x \in R(X)$. It follows from (3.1) that

$$(U_g f)_X(x) = \hat{U}_g f_X(x) \tag{3.2}$$

for every $x \in R(X)$. As in the proof of Theorem 3.2, $\hat{U}_g$ is a unitary operator on H_X satisfying $\hat{U}_{g_1 g_2} = \hat{U}_{g_1} \hat{U}_{g_2}$. Hence, $g \mapsto \hat{U}_g$ is a unitary representation of G on H_X.

We now give a simple, but important, example of a symmetry group on a catalog. We consider a physical system consisting of a particle of mass m moving in three-space $\mathbb{R}^3$. We then take as our sample space the phase space

$$\Omega = \mathbb{R}^6 = \{(\mathbf{q}, \mathbf{p})\colon \mathbf{q}, \mathbf{p} \in \mathbb{R}^3\}$$

There are two natural measurements Q, P defined by $Q(\mathbf{q}, \mathbf{p}) = \mathbf{q}$, $P(\mathbf{q}, \mathbf{p}) = \mathbf{p}$. On the fiber

$$Q^{-1}(\mathbf{q}) = \{(\mathbf{q}, \mathbf{p})\colon \mathbf{p} \in \mathbb{R}^3\}$$

we let Σ_Q^q be the usual Borel σ-algebra and take μ_Q^q to be Lebesgue measure. On the range $R(Q) = \mathbb{R}^3$ we let Σ_Q be the Borel σ-algebra and again take μ_Q to be Lebesgue measure. Similar constructions are employed for P. In this way Q, P are measurements and $\mathcal{A} = \{Q, P\}$ is a catalog on Ω. It is shown in Gudder (1988) that there exist many amplitudes on $\mathcal{A}$. Moreover, it is shown that this structure gives the same predictions as the usual nonrelativistic quantum mechanics.

We now define the isochronous Galilei group G on Ω. The elements of G are triplets $g = (\mathbf{a}, \mathbf{v}, R)$, where $\mathbf{a} \in \mathbb{R}^3$ represents a space translation, $\mathbf{v} \in \mathbb{R}^3$ a velocity boost, and $R \in SO(3)$ a rotation. The action of G on Ω is given by

$$g(\mathbf{q}, \mathbf{p}) = (\mathbf{a}, \mathbf{v}, R)(\mathbf{q}, \mathbf{p}) = (\mathbf{a} + R\mathbf{q}, m\mathbf{v} + R\mathbf{p})$$

It is easy to check that the group multiplication becomes

$$\begin{aligned} g_1 g_2 &= (\mathbf{a}_1, \mathbf{v}_1, R_1)(\mathbf{a}_2, \mathbf{v}_2, R_2) \\ &= (\mathbf{a}_1 + R_1\mathbf{a}_2, \mathbf{v}_1 + R_1\mathbf{v}_2, R_1 R_2) \end{aligned}$$

Moreover, it is straightforward to show that G is a symmetry group on $\mathcal{A}$.

4. SYSTEMS OF COVARIANCE

We now prove our main result. This result shows that $\hat{X}^S$ is a generalized system of covariance for the generalized unitary representation U_g.

Theorem 4.1. For every $g\in G$, $S\in\mathscr{S}(\mathscr{A})$, $X\in\mathscr{A}$, $A\in\Sigma_X$, we have

$$U_g^{-1}\hat{X}^{gS}(A)U_g=\hat{X}^S(g^{-1}A) \tag{4.1}$$

Proof. Suppose S is generated by $M\in\mathscr{M}(\mathscr{A})$. We first prove that

$$U_X^{gM}U_g=\hat{U}_gU_X^M \tag{4.2}$$

Letting $f\in M$, $x\in R(X)$, we have by (3.2) that

$$(U_X^{gM}U_gf)(x)=(U_gf)_X(x)=\hat{U}_gf_X(x)=(\hat{U}_gU_X^Mf)(x)$$

so (4.2) holds. We can extend (4.2) to S to obtain

$$U_X^{gS}U_g=\hat{U}_gU_X^S \tag{4.3}$$

We now show that

$$\hat{U}_gP_X^S=P_X^{gS}\hat{U}_g \tag{4.4}$$

Let $h\in U_X^SS$. Then $h=U_X^Sf$, $f\in S$, and by (4.3)

$$\hat{U}_gh=\hat{U}_gU_X^Sf=U_X^{gS}U_gf\in U_X^{gS}gS$$

Thus,

$$P_X^{gS}\hat{U}_gh=\hat{U}_gh=\hat{U}_gP_X^Sh$$

Now suppose $h\in(U_X^SS)^\perp$. Then $\hat{U}_gP_X^Sh=0$. Let $h'\in U_X^{gS}gS$. Then $h'=U_X^{gS}f'$ for some $f'\in gS$. Hence, by (4.3) we have

$$\hat{U}_g^{-1}h'=\hat{U}_{g^{-1}}U_X^{gS}f'=U_X^SU_{g^{-1}}f'\in U_X^SS$$

Therefore,

$$\langle\hat{U}_gh, h'\rangle=\langle h, \hat{U}_g^{-1}h'\rangle=0$$

so that $\hat{U}_gh\in(U_X^{gS}gS)^\perp$. Hence, $P_X^{gS}\hat{U}_gh=0$ and (4.4) holds.

We next show that

$$U_gV_X^S=V_X^{gS}\hat{U}_g \tag{4.5}$$

Applying (4.3), we have

$$U_g=V_X^{gS}\hat{U}_gU_X^S$$

Hence, from (4.4) we obtain

$$U_g V_X^S = V_X^{gS} \hat{U}_g P_X^S = V_X^{gS} P_X^{gS} \hat{U}_g = V_X^{gS} \hat{U}_g$$

so (4.5) holds. We now show that

$$\chi_A U_X^{gS} U_g = \hat{U}_g \chi_{g^{-1}A} U_X^S \tag{4.6}$$

Letting $f \in S$, we have by (4.3)

$$\begin{aligned}(\chi_A U_X^{gS} U_g f)(x) &= \chi_A(x)(U_X^{gS} U_g f)(x) = \chi_A(x)(\hat{U}_g U_X^S f)(x)\\ &= \chi_{g^{-1}A}(x_{g^{-1}})(U_X^S f)(x_{g^{-1}})\\ &= (\hat{U}_g \chi_{g^{-1}A} U_X^S f)(x)\end{aligned}$$

so (4.6) holds. Finally, applying (4.6) and (4.5) gives

$$\begin{aligned}\hat{X}^{gS}(A) U_g &= V_X^{gS} \chi_A U_X^{gS} U_g = V_X^{gS} \hat{U}_g \chi_{g^{-1}A} U_X^S\\ &= U_g V_X^S \chi_{g^{-1}A} U_X^S = U_g \hat{X}^S(g^{-1}A)\end{aligned}$$

The result now follows. ■

We say that G *leaves sectors invariant* if for every $S \in \mathcal{S}(\mathcal{A})$ we have $gS = S$. It is easy to see that G leaves sectors invariant if and only if $f \, s \, h$ implies $f \, s \, U_g h$ for every $g \in G$. This is equivalent to the following condition. If $f, h \in M$, for any $M \in \mathcal{M}(\mathcal{A})$, then for every X, $Y \in \mathcal{A}$, $g \in G$, we have

$$\int f_X(x) \bar{h}_X(x_{g^{-1}}) \, d\mu_X(x) = \int f_Y(y) \bar{h}_Y(y_{g^{-1}}) \, d\mu_Y(y)$$

Corollary 4.2. If G leaves sectors invariant, then

$$U_g^{-1} \hat{X}^S(A) U_g = \hat{X}^S(g^{-1}A) \tag{4.7}$$

for all $g \in G$, $S \in \mathcal{S}(\mathcal{A})$, $X \in \mathcal{A}$, $A \in \Sigma_X$.

Equation (4.7) is the usual condition for $\hat{X}^S$ to be a system of covariance for U_g and in this case U_g is a usual unitary representation of G. For $X \in \mathcal{A}$, $A \in \Sigma_X$, define the projection operator $Q_X(A)h = \chi_A h$ on H_X. Then $A \mapsto Q_X(A)$ is a projection-valued (PV) measure from Σ_X to H_X. We can now obtain a much simpler result than Theorem 4.1. Namely, for all $A \in \Sigma_X$

$$\hat{U}_g^{-1} Q_X(A) \hat{U}_g = Q_X(g^{-1}A) \tag{4.8}$$

To prove (4.8), letting $h \in H_X$, we have

$$\begin{aligned}(\hat{U}_g Q_X(g^{-1}A)h)(x) &= \chi_{g^{-1}A}(x_{g^{-1}}) h(x_{g^{-1}})\\ &= \chi_A(x) h(x_{g^{-1}}) = (Q_X(A) \hat{U}_g h)(x)\end{aligned}$$

Hence,

$$\hat{U}_g Q_X(g^{-1}A) = Q_X(A) \hat{U}_g$$

and (4.8) follows. This shows that Q_X is a system of imprimitivity for the unitary representation $\hat{U}_g$. However, (4.8) has a much weaker interpretation than (4.1). This is because (4.8) concerns the representation of a single measurement X on the Hilbert space H_X, while (4.1) represents all the measurements in $\mathscr{A}$ simultaneously on the Hilbert space S. Thus, in (4.8) a different Hilbert space H_X is used for each measurement X, while in (4.1) a single Hilbert space S is employed. In the latter case, various measurements can be compared or combined and this is impossible in the former case.

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